

Systems of reflected quasilinear stochastic PDEs in a convex domain

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Abstract: This paper presents existence and uniqueness results for reflected system of quasilinear stochastic partial differential equations in a convex domain D from \mathbb{R}^k . The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random signed regular measure satisfying the minimal Skorohod condition.

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1. Introduction

Stochastic partial differential equations (SPDEs) are a powerful tool to model various phenomena from biology to finance. They can be employed, for example, to describe the evolution of action potentials in the brain, or to model interest rates. They appeared also in phase transitions and front propagation in random media and with random normal velocities, filtering and stochastic control with partial observations, pathwise stochastic control theory, mathematical finance, etc.

It is well known now that BSDEs give a probabilistic interpretation for the solution of a class of semi-linear PDEs. By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs in short) [19] (see also [3], [15]), which can be seen as Feynman-Kac representation of SPDEs and provide a

powerful tool for probabilistic numerical schemes [1] for such SPDEs. Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions, namely, Sobolev's solutions [6, 8, 9, 20, 22], and stochastic viscosity solutions [12, 13, 14, 4, 5].

Given a convex domain D in \mathbb{R}^k , we are concerned with the study of weak solutions of the reflection problem for a system of quasilinear SPDE with values in the domain D . In the deterministic case, this class has been studied by Moser (1961), and Serrin Aronson (1967). In our case, we consider the same class of PDE but perturbed by a nonlinear noise directed by a finite -dimensional Brownian motion. We are interested in the study of the reflection problem for quasilinear SPDEs and which is defined as a pair (u, ν) , where ν is a random and regular measure $u \in \mathbf{L}^2(\Omega \times [0, T]; H^1(\mathbb{R}^d))$ satisfy the following relations:

$$\left\{ \begin{array}{l} (i) \quad u(t, x) \in \bar{D}, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\ (ii) \quad du(t, x) + \left(\partial_i [a_{ij}(t, x) \partial_j u(t, x) + g_i(t, x, u(t, x), \nabla u(t, x))] + f(t, x, u(t, x), \nabla u(t, x)) \right) dt \\ \quad + h(t, x, u(t, x), \nabla u(t, x)) \cdot d\overleftarrow{W}_t = -\nu(dt, dx), \text{a.s.} \\ (iii) \quad \nu(u \notin \partial D) = 0, \text{a.s.}, \\ (iv) \quad u(T, x) = \Phi(x), \quad dx - \text{a.e.} \end{array} \right. \quad (1.1)$$

where a is a time-dependent symmetric uniformly elliptic measurable matrix, f, h, g are non-linear random functions measurable in (t, x) and Lipschitz in (y, z) . The differential term with $d\overleftarrow{W}_t$ refers to the backward stochastic integral with respect to a l -dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, (W_t)_{t \geq 0})$.

In the one dimensional case, Matoussi and Stoica [16] have proved an existence and uniqueness result for the obstacle problem of quasilinear stochastic PDE. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (BDSDE in short). They have also proved that the solution is a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying the minimal Skohorod condition. In particular, they gave for the regular measure ν a probabilistic interpretation in terms of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

An essential ingredient in the study of the quasilinear part is the probabilistic representation of the divergence term obtained in [21] as well as the doubly stochastic representation corresponding to the divergence term of the stochastic PDE in [16].

Our main contribution in this work is to establish a priori estimates based on extension of Ito's formula for the solution of generalized BDSDES associated to the quasilinear SPDEs.

The paper is structured in the following way: We introduce in Section 2 several notations and hypothesis that will be used throughout the paper. Then, a weak formulation for the system of quasilinear SPDEs is given in Definition 2.2. The main results of this paper are presented in Section 3. Indeed, the existence and uniqueness result of the weak solution for quasilinear RSPDEs are established by using a penalization method. In particular, an extension of Ito's formula and a priori estimates are proved. Moreover, a probabilistic representation of this solution is proven via the solution of generalized markovian RBDSDEs. In the Appendix, technical lemmas for the existence of the solution of the Reflected BDSDEs and SPDEs in a convex domain are given.

2. Weak solution of quasilinear SPDE in a convex domain

The euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $k \times k$ matrix A , we define $\|A\| = \sqrt{\text{Tr}AA^\top}$. In what follows let us fix a positive number $T > 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability product space, and let $\{W_s, 0 \leq s \leq T\}$ and $\{B_s, 0 \leq s \leq T\}$ be two mutually independent standard Brownian motion processes, with values respectively in \mathbb{R}^l and in \mathbb{R}^d . For each $t \in [0, T]$, we define

$$\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^W \vee \mathcal{N}$$

where $\mathcal{F}_t^B = \sigma\{B_r, 0 \leq r \leq t\}$, $\mathcal{F}_{t,T}^W = \sigma\{W_r - W_t, t \leq r \leq T\}$ and \mathcal{N} the class of \mathbb{P} null sets of \mathcal{F} . Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration.

2.1. Transformation of the equation

We note that we can reduce the study of our problem (1.1) using the transformation given in Matoussi and Stoica [16] (Remark 1, p. 1157). Indeed, we denote by $L = \sum_{i,j} \partial_i a_{ij} \partial_j$ the elliptic operator such that

$$\lambda |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi^i \xi^j \leq \Lambda |\xi|^2.$$

then the time change $t \rightarrow \frac{1}{2\Lambda} t'$ yields to one correspondence between the solutions u of the equation

$$\begin{aligned} du(t, x) + [Lu(t, x) + \text{div}(g(t, x, u(t, x), \nabla u(t, x))) + f(t, x, u(t, x), \nabla u(t, x))] dt \\ + h(t, x, u(t, x), \nabla u(t, x)) \cdot d\overleftarrow{W}_t = 0, \end{aligned} \quad (2.1)$$

over $[0, T]$ and the solutions $\hat{u}(t, \cdot) = u(\frac{1}{2\Lambda}t, \cdot)$ satisfying the equation

$$\begin{aligned} d\hat{u}(t, x) + \left[\frac{1}{2}\Delta\hat{u}(t, x) + \operatorname{div}(\hat{g}(t, x, \hat{u}(t, x), \nabla\hat{u}(t, x))) + \hat{f}(t, x, \hat{u}(t, x), \nabla\hat{u}(t, x)) \right] dt \\ + \hat{h}(t, x, \hat{u}(t, x), \nabla\hat{u}(t, x)) \cdot d\overleftarrow{W}_t = 0, \end{aligned} \quad (2.2)$$

over the interval $[0, 2\Lambda T]$, with the transformed coefficients

$$\begin{aligned} \hat{f}(t, x, y, z) &:= \frac{1}{2\Lambda} f\left(\frac{1}{2\Lambda}t, x, y, z\right) \quad , \quad \hat{h}(t, x, y, z) := \frac{1}{(2\Lambda)^{1/2}} h\left(\frac{1}{2\Lambda}t, x, y, z\right) \\ \hat{g}(t, x, y, z) &:= \frac{1}{2\Lambda} \left(g\left(\frac{1}{2\Lambda}t, x, y, z\right) + \gamma(x)z \right) \quad , \quad \gamma = \Lambda I - a. \end{aligned}$$

From now, we focus our study on solving a system of reflected quasilinear stochastic PDEs of this form

$$\begin{aligned} du(t, x) + \left[\frac{1}{2}\Delta u(t, x) + \operatorname{div}(g(t, x, u(t, x), \nabla u(t, x))) + f(t, x, u(t, x), \nabla u(t, x)) \right] dt \\ + h(t, x, u(t, x), \nabla u(t, x)) \cdot d\overleftarrow{W}_t = 0. \end{aligned}$$

Our main interest is the study of weak solutions to the reflection problem for multidimensional SPDEs in a convex domain D in \mathbb{R}^k . We consider the solution of the reflection problem for the quasilinear SPDEs (1.1) as a pair (u, ν) , where ν is a random regular measure and $u \in \mathbf{L}^2(\Omega \times [0, T]; H^1(\mathbb{R}^d))$ satisfies the following relations:

$$\left\{ \begin{array}{l} (i) \quad u(t, x) \in \bar{D}, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\ (ii) \quad du(t, x) + \left[\frac{1}{2}\Delta u(t, x) + f(t, x, u(t, x), \nabla u(t, x)) + \operatorname{div}(g(t, x, u(t, x), \nabla u(t, x))) \right] dt \\ \quad + h(t, x, u(t, x), \nabla u(t, x)) \cdot d\overleftarrow{W}_t = -\nu(dt, dx), \text{a.s.} \\ (iii) \quad \nu(u \notin \partial D) = 0, \text{a.s.}, \\ (iv) \quad u(T, x) = \Phi(x), \quad dx - \text{a.e.} \end{array} \right. \quad (2.3)$$

ν is a random measure which acts only when the process u reaches the boundary of the domain D . The rigorous sense of the relation (iii) will be based on the probabilistic representation of the measure ν in term of the bounded variation processes K , a component of the associated solution of the reflected BDSDE in the domain D . This problem is well known as a Skorohod problem for SPDEs.

2.2. Notations and Hypothesis

Let us first introduce some functional spaces:

- $C_{l,b}^n(\mathbb{R}^p, \mathbb{R}^q)$ the set of C^n -functions which grow at most linearly at infinity and whose partial

derivatives of order less than or equal to n are bounded.

- $\mathbf{L}_\rho^2(\mathbb{R}^d)$ will be a Hilbert with the inner product,

$$(u, v)_\rho = \int_{\mathbb{R}^d} u(x) v(x) \rho(x) dx, \quad \|u\|_2 = \left(\int_{\mathbb{R}^d} u^2(x) \rho(x) dx \right)^{\frac{1}{2}},$$

where ρ is the weight function that satisfy the following conditions:

Assumption 2.1.

- ρ is a continuous positive function.
- ρ is integrable and $\frac{1}{\rho}$ is locally integrable.

In general, we shall use for the usual L^2 -scalar product

$$(u, v) = \int_{\mathbb{R}^d} u(x) v(x) dx,$$

where u, v are measurable functions defined in \mathbb{R}^d and $uv \in \mathbf{L}^1(\mathbb{R}^d)$.

Our evolution problem will be considered over a fixed time interval $[0, T]$ and the norm for an element of $\mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$ will be denoted by

$$\|u\|_{2,2} = \left(\int_0^T \int_{\mathbb{R}^d} |u(t, x)|^2 \rho(x) dx dt \right)^{\frac{1}{2}}.$$

We assume the following hypotheses :

Assumption 2.2. $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^k$ belongs to $\mathbf{L}_\rho^4(\mathbb{R}^d)$ and $\Phi(x) \in \bar{D}$ a.e. $\forall x \in \mathbb{R}^d$;

Assumption 2.3. (i) $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$, $h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times l}$ and $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times d}$ are measurable in (t, x, y, z) and satisfy $f^0, h^0 \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$ where $f_t^0(x) := f(t, x, 0, 0)$, $h_t^0 := h(t, x, 0, 0)$ and $g_t^0(x) := g(t, x, 0, 0)$.

(ii) There exist constants $c > 0, 0 < \alpha < 1$ and $0 < \beta < 1$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^d$; $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| &\leq c(|y_1 - y_2| + \|z_1 - z_2\|) \\ \|h(t, x, y_1, z_1) - h(t, x, y_2, z_2)\| &\leq c|y_1 - y_2| + \beta\|z_1 - z_2\| \\ \|g(t, x, y_1, z_1) - g(t, x, y_2, z_2)\| &\leq c|y_1 - y_2| + \alpha\|z_1 - z_2\|. \end{aligned}$$

(iii) The contract property: $\alpha + \frac{\beta^2}{2} < \frac{1}{2}$.

Assumption 2.4. (i) There exist $c > 0$ and $0 \leq \gamma < 1$ such that for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$h h^\top(t, x, y, z) \leq c(Id_{\mathbb{R}^k} + yy^\top) + \gamma zz^\top.$$

(ii) f, h and g are uniformly bounded in (x, y, z) .

Remark 2.1. 1. The Assumption 2.4 (i) and (ii) are needed to prove the uniform L^4 -estimate for (Y^n, Z^n) solution of BDSDE (3.7) (see estimate (A.6) in the Appendix B). This is crucial for our proof of the fundamental lemma 3.2.

2. The Assumption 2.4 (ii) is only added for simplicity and it can be removed by standard technics of BSDEs. The natural condition instead of (ii) is f^0 and h^0 in $\mathbf{L}^4(\Omega, \mathcal{F}, \mathbb{P})$.

Since the domain D is convex we need to recall some properties that we will use later. Let ∂D denotes the boundary of D and $\pi(x)$ the projection of $x \in \mathbb{R}^k$ on D . We have the following properties:

$$(x' - x)^\top (x - \pi(x)) \leq 0, \quad \forall x \in \mathbb{R}^d, \quad \forall x' \in \bar{D} \quad (2.4)$$

$$(x' - x)^\top (x - \pi(x)) \leq (x' - \pi(x'))^\top (x - \pi(x)), \quad \forall x, x' \in \mathbb{R}^k \quad (2.5)$$

$$\exists a \in D, \gamma > 0, \text{ such that } (x - a)^\top ((x - \pi(x))) \geq \gamma |x - \pi(x)|, \quad \forall x \in \mathbb{R}^k. \quad (2.6)$$

For $x \in \partial D$, we denote by $n(x)$ the set of outward normal unit vectors at the point x .

One can find all these results in Menaldi [18], page 737.

2.3. The measures \mathbb{P}^m

The operator $\partial_t + \frac{1}{2}\Delta$, which represents the main linear part in the equation (2.3), is probabilistically interpreted by the Brownian motion in \mathbb{R}^d . We shall view the Brownian motion as a Markov process and therefore we next introduce some detailed notation for it. The sample space is $\Omega' = \mathcal{C}([0, \infty); \mathbb{R}^d)$, the canonical process $(B_t)_{t \geq 0}$ is defined by $B_t(\omega) = \omega(t)$, for any $\omega \in \Omega'$, $t \geq 0$ and the shift operator, $\theta_t : \Omega' \rightarrow \Omega'$, is defined by $\theta_t(\omega)(s) = \omega(t + s)$, for any $s \geq 0$ and $t \geq 0$. The canonical filtration $\mathcal{F}_t^B = \sigma(B_s; s \leq t)$ is completed by the standard procedure with respect to the probability measures produced by the transition function

$$P_t(x, dy) = q_t(x - y)dy, \quad t > 0, \quad x \in \mathbb{R}^d,$$

where $q_t(x) = (2\pi t)^{-\frac{d}{2}} \exp(-|x|^2/2t)$ is the gaussian density. Thus we get a continuous Hunt process $(\Omega', B_t, \theta_t, \mathcal{F}, \mathcal{F}_t^0, \mathbb{P}^x)$. We shall also use the backward filtration of the future events $\mathcal{F}'_t = \sigma(B_s; s \geq t)$ for $t \geq 0$. \mathbb{P}^0 is the Wiener measure, which is supported by the set $\Omega'_0 = \{\omega \in \Omega', w(0) = 0\}$. We also set $\Pi_0(\omega)(t) = \omega(t) - \omega(0)$, $t \geq 0$, which defines a map $\Pi_0 : \Omega' \rightarrow \Omega'_0$. Then $\Pi = (B_0, \Pi_0) : \Omega' \rightarrow \mathbb{R}^d \times \Omega'_0$ is a bijection. For each probability measure on \mathbb{R}^d , the probability \mathbb{P}^μ of the Brownian motion started with the initial distribution μ is given by

$$\mathbb{P}^\mu = \Pi^{-1}(\mu \otimes \mathbb{P}^0).$$

In particular, for the Lebesgue measure in \mathbb{R}^d , which we denote by $m = dx$, we have

$$\mathbb{P}^m = \Pi^{-1}(dx \otimes \mathbb{P}^0).$$

2.4. Weak formulation for a solution of Stochastic PDEs

The space of test functions which we employ in the definition of weak solutions of the evolution equations (1.1) is $\mathcal{D}_T = \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}_c^\infty(\mathbb{R}^d)$, where

- $\mathcal{C}^\infty([0, T])$ denotes the space of real functions which can be extended as infinitely differentiable functions in the neighborhood of $[0, T]$,
- $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is the space of infinite differentiable functions with compact supports in \mathbb{R}^d .

We denote by \mathcal{H}_T the space of $\mathcal{F}_{t,T}^W$ -progressively measurable processes (u_t) with values in the weighted Sobolev space $H_\rho^1(\mathbb{R}^d)$ where

$$H_\rho^1(\mathbb{R}^d) := \{v \in \mathbf{L}_\rho^2(\mathbb{R}^d) \mid \nabla v \in \mathbf{L}_\rho^2(\mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T}^2 = \mathbb{E} \left[\sup_{0 \leq s \leq T} \|u_s\|_2^2 + \int_0^T \int_{\mathbb{R}^d} |\nabla u_s(x)|^2 ds \rho(x) dx \right],$$

where we denote the gradient by $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$. Here, the derivative is defined in the weak sense (Sobolev sense).

Definition 2.1 (Weak solution of quasilinear SPDE without reflection). *We say that $u \in \mathcal{H}_T$ is a Sobolev solution of SPDE (1.1) if the following relation holds, for each $\varphi \in \mathcal{D}_T$,*

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} [u(s, x) \partial_s \varphi(s, x) + \frac{1}{2} \nabla u(s, x) \nabla \varphi(s, x)] dx ds + \int_{\mathbb{R}^d} [u(t, x) \varphi(t, x) - \Phi(x) \varphi(T, x)] dx \\ &= \int_t^T \int_{\mathbb{R}^d} [f(s, x, u(s, x), \nabla u(s, x)) \varphi(s, x) - g(s, x, u(s, x), \nabla u(s, x)) \nabla \varphi(s, x)] dx ds \\ &+ \int_t^T \int_{\mathbb{R}^d} h(s, x, u(s, x), \nabla u(s, x)) \varphi(s, x) dx d\overleftarrow{W}_s. \end{aligned} \quad (2.7)$$

We denote by $u := \mathcal{U}(\Phi, f, g, h)$ the solution of SPDEs with data (Φ, f, g, h) .

The existence and uniqueness of weak solution for SPDEs (2.7) is ensured by Theorem 8 in Denis and Stoica [6].

We now precise the definition of weak solutions for the reflected quasilinear SPDE (2.3):

Definition 2.2 (Weak solution of quasilinear RSPDE). *We say that $(u, \nu) := (u^i, \nu^i)_{1 \leq i \leq k}$ is the weak solution of the reflected SPDE (2.3) associated to (Φ, f, g, h) , if for each $1 \leq i \leq k$*

(i) $\|u\|_{\mathcal{H}_T} < \infty$, $u_t(x) \in \bar{D}$, $dx \otimes dt \otimes d\mathbb{P}$ a.e., and $u(T, x) = \Phi(x)$.

(ii) ν^i is a signed Random measure on $[0, T] \times \mathbb{R}^d$ such that:

a) ν^i is adapted in the sense that for any measurable function $\psi : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and for each $s \in [t, T]$, $\int_s^T \int_{\mathbb{R}^d} \psi(r, x) \nu^i(dr, dx)$ is $\mathcal{F}_{s,T}^W$ -measurable.

b) $\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \rho(x) |\nu^i|(dt, dx) \right] < \infty$.

(iii) for every $\varphi \in \mathcal{D}_T$

$$\begin{aligned}
& \int_t^T \int_{\mathbb{R}^d} [u^i(s, x) \partial_s \varphi(s, x) + \frac{1}{2} \nabla u^i(s, x) \nabla \varphi(s, x)] dx ds + \int_{\mathbb{R}^d} [u^i(t, x) \varphi(t, x) - \Phi^i(x) \varphi(T, x)] dx \\
&= \int_t^T \int_{\mathbb{R}^d} [f(s, x, u(s, x), \nabla u(s, x)) \varphi(s, x) - g(s, x, u(s, x), \nabla u(s, x)) \nabla \varphi(s, x)] dx ds \\
&+ \int_t^T \int_{\mathbb{R}^d} h(s, x, u(s, x), \nabla u(s, x)) \varphi(s, x) dx d\overleftarrow{W}_s + \int_t^T \int_{\mathbb{R}^d} \varphi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx).
\end{aligned} \tag{2.8}$$

For the sake of simplicity we will omit in the sequel the subscript i .

We are concerned with solving our problem by developing a stochastic flow method which was first introduced in Kunita [10], [11] and Bally, Matoussi [3]. We recall that $\{B_{t,s}(x), t \leq s \leq T\}$ is the diffusion process starting from x at time t and is the strong solution of the equation:

$$B_{t,s}(x) = x + (B_s - B_t). \tag{2.9}$$

Moreover the inverse of the flow satisfies the following backward SDE

$$B_{t,s}^{-1}(y) = y - (B_s - B_t). \tag{2.10}$$

for any $t < s$.

3. Existence and uniqueness of the system of quasilinear SPDEs

In this section, we will establish the existence and uniqueness result of the weak solution for quasilinear RSPDEs (2.3) by using a penalization method. Then, we prove a probabilistic representation of this solution via the solution of generalized markovian RBDSEs. The first main result of this section is the following:

Theorem 3.1. *Let Assumptions 2.2-2.4 hold. Then there exists a unique weak solution (u, ν) of the reflected SPDE (2.3) associated to (Φ, f, g, h) .*

Moreover, we establish the following probabilistic representation for (u, ν) :

Theorem 3.2. *Let Assumptions 2.2-2.4 hold. Then $u(t, x) := Y_t^{t,x}$, $dt \otimes d\mathbb{P} \otimes -a.e.$, and*

$$Y_s^{t,x} = u(s, B_{t,s}(x)), \quad Z_s^{t,x} = (\nabla u)(s, B_{t,s}(x)), \quad ds \otimes d\mathbb{P} \otimes -a.e.. \tag{3.1}$$

Furthermore, for every measurable bounded and positive functions φ and ψ ,

$$\int_{\mathbb{R}^d} \int_t^T \varphi(s, B_s^{-1}) \psi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_s) dK_s^i dx, \quad a.s.. \tag{3.2}$$

where $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$ is the solution of this class of markovian RBDSDE

$$\left\{ \begin{array}{l} (i) \ Y_s^{t,x} = \Phi(B_{t,T}(x)) + \int_s^T f(r, B_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T h(r, B_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r + K_T^{t,x} - K_s^{t,x} \\ \quad + \frac{1}{2} \int_t^T g(r, B_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) * dB_r - \int_s^T Z_r^{t,x} dB_r, \ \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.}, \ \forall s \in [t, T] \\ (ii) \ Y_s^{t,x} \in \bar{D} \quad \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.} \\ (iii) \ \int_0^T (Y_s^{t,x} - v_s(B_{t,s}(x)))^* dK_s^{t,x} \leq 0., \ \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.}, \\ \text{for any continuous } \mathcal{F}_t\text{-random function } v : [0, T] \times \Omega \times \mathbb{R}^d \longrightarrow \bar{D}. \end{array} \right. \quad (3.3)$$

3.1. Proof of existence by penalization method

The existence of a solution will be proved by penalization method. For $n \in \mathbb{N}$, we consider the penalized system of quasilinear SPDE:

$$\left\{ \begin{array}{l} du^n(s, x) + [\frac{1}{2} \Delta u^n(s, x) + f(s, x, u^n(s, x), \nabla u^n(s, x)) + \text{div}(g(s, x, u^n(s, x), \nabla u^n(s, x)))] ds \\ \quad + h(s, x, u^n(s, x), \nabla u^n(s, x)) \cdot d\overleftarrow{W}_s - n(u^n(s, x) - \pi(u^n(s, x))) ds = 0, \\ u^n(T, x) = \Phi(x) \end{array} \right. \quad (3.4)$$

From Denis and Stoica [6] (Theorem 8), we know that the above equation admits a unique weak solution of the SPDE (Φ, f^n, h, g) (2.7), with $f^n(t, x, y) = f(t, x, y, z) - n(y - \pi(y))$, i.e. for every $\varphi \in \mathcal{D}_T$.

$$\begin{aligned} & \int_t^T [(u^n(s, \cdot), \partial_s \varphi(s, \cdot)) + \frac{1}{2} (\nabla u^n(s, \cdot), \nabla \varphi(s, \cdot))] ds + (u^n(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) \\ &= \int_t^T [(f^n(s, \cdot, u^n(s, \cdot), \nabla u^n(s, \cdot)), \varphi(s, \cdot)) + (g(s, \cdot, u^n(s, \cdot), \nabla u^n(s, \cdot)), \varphi(s, \cdot))] ds \\ &+ \int_t^T (h(s, \cdot, u^n(s, \cdot), \nabla u^n(s, \cdot)), \varphi(s, \cdot)) d\overleftarrow{W}_s. \end{aligned} \quad (3.5)$$

We are going to show that $(u^n)_{n \geq 1}$ is a Cauchy sequence in \mathcal{H}_T with the help of doubly backward stochastic differential equation technics. Denote

$$\begin{aligned} Y_s^{n,t,x} &= u^n(s, B_{t,s}(x)) \quad , \quad Z_s^{n,t,x} = \nabla u^n(s, B_{t,s}(x)) \\ K_s^{n,t,x} &= -n \int_0^s [u^n(r, B_{t,r}(x)) - \pi(u^n(r, B_{t,r}(x)))] dr. \end{aligned} \quad (3.6)$$

By following the representation in Matoussi and Stoica [16] (Theorem 1 p.1148), we see that $(Y^{n,t,x}, Z^{n,t,x})$ solves the following relation \mathbb{P}^m -a.s.:

$$\begin{aligned} Y_s^{n,t,x} &= \Phi(B_{t,T}(x)) + \int_s^T f(r, B_{t,r}(x), Y_r^{n,t,x}, Z_r^{n,t,x}) dr + \int_s^T h(r, B_{t,r}(x), Y_r^{n,t,x}, Z_r^{n,t,x}) d\overleftarrow{W}_r \\ &+ K_T^{n,t,x} - K_s^{n,t,x} + \frac{1}{2} \int_s^T g(r, B_{t,r}(x), Y_r^{n,t,x}, Z_r^{n,t,x}) * dB_r - \int_s^T Z_r^{n,t,x} dB_r. \end{aligned} \quad (3.7)$$

Remark 3.1. The subscripts (t, x) will often be dropped for notational simplicity if the context is clear and the notations $B_t = B_{t,s}(x)$ and $B_t^{-1} = B_{t,s}^{-1}(y)$ will be frequently used throughout.

To prove that $(Y^n, Z^n), n \geq 1$ is a Cauchy sequence, we need to prepare a number of results. We start with the following lemma:

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\forall n \in \mathbb{N} \quad \mathbb{E} \mathbb{E}^m \left[\int_0^T d^2(Y_s^n, D) ds \right] \leq C \left(\frac{1}{n} + \frac{1}{n^2} \right). \quad (3.8)$$

Proof. We apply the double stochastic Itô's formula extended in Matoussi and Stoica (Corollary 1 and Remark 2 in [16] p.1158) to $\rho(u^n(t, B_t)) = \rho(Y_t^n) = d^2(Y_t^n, D) = |Y_t^n - \pi(Y_t^n)|^2$ to obtain

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\rho(Y_s^n)] ds &= \rho(\Phi(B_T)) + \int_t^T (\nabla \rho(Y_s^n))^\top f(s, B_s, Y_s^n, Z_s^n) ds \\ &\quad - \int_t^T (\nabla \rho(Y_s^n))^\top Z_s^n dB_s + \int_t^T (\nabla \rho(Y_s^n))^\top h(s, B_s, Y_s^n, Z_s^n) d\bar{W}_s \\ &\quad + \frac{1}{2} \int_t^T \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) \text{Hess}\rho(Y_s^n)] ds - 2n \int_t^T (Y_s^n - \pi(Y_s^n))^\top (Y_s^n - \pi(Y_s^n)) ds \\ &\quad + \frac{1}{2} \sum_{i=1}^k \int_t^T \frac{\partial \rho}{\partial y_i}(Y_s^n) g^i(s, B_s, Y_s^n, Z_s^n) * dB_s - \sum_{i=1}^k \int_t^T \langle \nabla[\frac{\partial \rho}{\partial y_i}(u^n(s, B_s))], g^i(s, B_s, Y_s^n, Z_s^n) \rangle ds, \end{aligned} \quad (3.9)$$

where ∇ is taken for the argument $x \in \mathbb{R}^d$. Now, for the last term we have

$$\begin{aligned} - \sum_{i=1}^k \int_t^T \langle \nabla[\frac{\partial \rho}{\partial y_i}(u^n(s, B_s))], g^i(s, Y_s^n, Z_s^n) \rangle ds &= - \sum_{i=1}^k \int_t^T \sum_{l=1}^d \frac{\partial}{\partial x_l} [\frac{\partial \rho}{\partial y_i}(u^n(s, B_s))] g^{il}(s, Y_s^n, Z_s^n) ds \\ &= - \sum_{i=1}^k \int_t^T \sum_{l=1}^d \sum_{j=1}^k [\frac{\partial^2 \rho}{\partial y_j \partial y_i}(u^n(s, B_s)) \frac{\partial u^{n,j}(B_s)}{\partial x_l}] g^{il}(s, Y_s^n, Z_s^n) ds \\ &= - \sum_{i=1}^k \int_t^T \sum_{l=1}^d \sum_{j=1}^k [\frac{\partial^2 \rho}{\partial y_j \partial y_i}(Y_s^n) Z_s^{n,jl}] g^{il}(s, Y_s^n, Z_s^n) ds \\ &= - \sum_{l=1}^d \int_t^T \langle \text{Hess}\rho(Y_s^n) Z_s^{n,\cdot l}, g^{\cdot l}(s, Y_s^n, Z_s^n) \rangle ds, \end{aligned} \quad (3.10)$$

where $Z_s^{n,\cdot l}, g^{\cdot l}$ stands for the column vector. Noting that

$$\begin{aligned} &|\langle \text{Hess}\rho(Y_s^n) Z_s^{n,\cdot l}, g^{\cdot l}(s, B_s, Y_s^n, Z_s^n) \rangle| \\ &\leq \langle \text{Hess}\rho(Y_s^n) Z_s^{n,\cdot l}, Z_s^{n,\cdot l} \rangle^{\frac{1}{2}} \langle \text{Hess}\rho(Y_s^n) g^{\cdot l}(s, B_s, Y_s^n, Z_s^n), g^{\cdot l}(s, B_s, Y_s^n, Z_s^n) \rangle^{\frac{1}{2}}, \end{aligned}$$

it follows that

$$\begin{aligned} - \sum_{i=1}^k \int_t^T \langle \nabla[\frac{\partial \rho}{\partial y_i}(u^n(s, B_s))], g^i(s, B_s, Y_s^n, Z_s^n) \rangle ds \\ \leq \frac{1}{4} \sum_{l=1}^d \int_t^T \langle \text{Hess}\rho(Y_s^n) Z_s^{n,\cdot l}, Z_s^{n,\cdot l} \rangle ds + C \sum_{l=1}^d \int_t^T \langle \text{Hess}\rho(Y_s^n) g^{\cdot l}(s, Y_s^n, Z_s^n), g^{\cdot l}(s, B_s, Y_s^n, Z_s^n) \rangle ds \\ \leq \frac{1}{4} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\rho(Y_s^n)] ds + C \int_t^T \text{trace}[gg^\top \text{Hess}\rho(Y_s^n)] ds. \end{aligned} \quad (3.11)$$

Since $\Phi(B_T) \in \bar{D}$ a.s., we have that $\rho(\Phi(B_T)) = 0$. Substituting (3.11) into (3.9) and taking into account the boundedness of h , g and the Hessian of ρ we obtain that

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{4} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\rho(Y_s^n)] ds + 2n \int_t^T d^2(Y_s^n, D) ds \\ \leq 2 \int_t^T (\rho(Y_s^n))^{1/2} |f(s, B_s, Y_s^n, Z_s^n)| ds - 2 \int_t^T (Y_s^n - \pi(Y_s^n))^\top Z_s^n dB_s \\ + 2 \int_t^T (Y_s^n - \pi(Y_s^n))^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s - \frac{1}{2} \sum_{i=1}^k \int_t^T \frac{\partial \rho}{\partial y_i} g^i(s, B_s, Y_s^n, Z_s^n) * dB_s + C. \end{aligned} \quad (3.12)$$

Now the inequality $2ab \leq a^2 + b^2$ with $a = \sqrt{\frac{n}{2}\rho(Y_s^n)}$ yields

$$(\rho(Y_s^n))^{1/2} |f(s, B_s, Y_s^n, Z_s^n)| \leq \frac{n}{4} \rho(Y_s^n) + \frac{1}{n} |f(s, B_s, Y_s^n, Z_s^n)|^2.$$

Then it follows that,

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{4} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\rho(Y_s^n)] ds + \frac{3n}{2} \int_t^T d^2(Y_s^n, D) ds \\ \leq 2 \int_t^T \frac{1}{n} |f(s, B_s, Y_s^n, Z_s^n)|^2 ds - 2 \int_t^T (Y_s^n - \pi(Y_s^n))^\top Z_s^n dB_s \\ + 2 \int_t^T (Y_s^n - \pi(Y_s^n))^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s - \frac{1}{2} \sum_{i=1}^k \int_t^T \frac{\partial \rho}{\partial y_i} g^i(s, B_s, Y_s^n, Z_s^n) * dB_s + C. \end{aligned} \quad (3.13)$$

By taking expectation and using the boundedness of f and the fact that under the measure \mathbb{P}^m the forward-backward integral $\int \frac{\partial \rho}{\partial y_i} g^i(s, B_s, Y_s^n, Z_s^n) * dB_s$ as well the other stochastic integrals with respect to the brownian terms have null expectation under $\mathbb{P} \otimes \mathbb{P}^m$, we have for all $0 \leq t \leq T$

$$\mathbb{E}\mathbb{E}^m[\rho(Y_t^n)] + \frac{1}{4} \mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\rho(Y_s^n)] ds \right] + \frac{3n}{2} \mathbb{E}\mathbb{E}^m \left[\int_t^T d^2(Y_s^n, D) ds \right] \leq C \left(1 + \frac{1}{n} \right). \quad (3.14)$$

Hence, we deduce that

$$\forall n \in \mathbb{N} \quad \mathbb{E}\mathbb{E}^m \left[\int_0^T d^2(Y_s^n, D) ds \right] \leq C \left(\frac{1}{n} + \frac{1}{n^2} \right).$$

□

In order to prove the strong convergence of the sequence (Y^n, Z^n, K^n) , we shall need the following result.

Lemma 3.2 (The essential step).

$$\mathbb{E}\mathbb{E}^m \left[\sup_{0 \leq t \leq T} (d(Y_t^n, D))^4 \right] \xrightarrow{n \rightarrow +\infty} 0. \quad (3.15)$$

Proof. We denote by $\rho(x) = d^2(x, D)$ and $\varphi(x) = \rho^2(x)$. By applying the double stochastic Itô's

formula to $\varphi(Y_t^n) = d^4(Y_t^n, D)$, we obtain that

$$\begin{aligned} \varphi(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\varphi(Y_s^n)] ds &= \varphi(\Phi(B_T)) + \int_t^T (\nabla\varphi(Y_s^n))^\top f(s, B_s, Y_s^n, Z_s^n) ds \\ &\quad - \int_t^T (\nabla\varphi(Y_s^n))^\top Z_s^n dB_s + \int_t^T (\nabla\varphi(Y_s^n))^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad + \frac{1}{2} \int_t^T \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)] ds - n \int_t^T \nabla\varphi(Y_s^n)^\top (Y_s^n - \pi(Y_s^n)) ds \\ &\quad + \frac{1}{2} \int_t^T (\nabla\varphi(Y_s^n))^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s - \sum_{i=1}^k \int_t^T \langle \nabla \frac{\partial\varphi}{\partial y_i}(u^n(s, \cdot)), g^i(s, B_s, u^n(s, \cdot), \nabla u^n(s, \cdot)) \rangle (B_s) ds. \end{aligned} \quad (3.16)$$

Using the similar arguments leading to the proof of (3.11), we obtain

$$\begin{aligned} - \sum_{i=1}^k \int_t^T \langle \nabla [\frac{\partial\varphi}{\partial y_i}(u^n(s, B_s))], g^i(s, B_s, Y_s^n, Z_s^n) \rangle ds \\ \leq \frac{1}{4} \sum_{l=1}^d \int_t^T \langle \text{Hess}\varphi(Y_s^n) Z_s^{n,l}, Z_s^{n,l} \rangle ds + C \sum_{l=1}^d \int_t^T \langle \text{Hess}\varphi(Y_s^n) g^l(s, B_s, Y_s^n, Z_s^n), g^l(s, B_s, Y_s^n, Z_s^n) \rangle ds \\ \leq \frac{1}{4} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\varphi(Y_s^n)] ds + C \int_t^T \text{trace}[gg^\top \text{Hess}\varphi(Y_s^n)] ds. \end{aligned} \quad (3.17)$$

Since $\Phi(B_T) \in \bar{D}$ a.s., we have that $\varphi(\Phi(B_T)) = 0$ and it is easy to see that

$$\nabla\varphi(x) = 2\rho(x)\nabla\rho(x) = 4\rho(x)(x - \pi(x)) \quad (3.18)$$

$$\text{Hess}\varphi(x) = 2\nabla\rho(x) \otimes \nabla\rho(x) + 2\rho(x)\text{Hess}\rho(x). \quad (3.19)$$

Combining (3.16), (3.17) together it follows that

$$\begin{aligned} \varphi(Y_t^n) + \frac{1}{4} \int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\varphi(Y_s^n)] ds &\leq 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^\top f(s, B_s, Y_s^n, Z_s^n) ds \\ &\quad - 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^\top Z_s^n dB_s + 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad - \frac{1}{2} \int_t^T (\nabla\varphi)^\top(Y_s^n) g(s, B_s, Y_s^n, Z_s^n) * dB_s - 4n \int_t^T \rho^2(Y_s^n) ds \\ &\quad + \frac{1}{2} \int_t^T \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)] ds + C \int_t^T \text{trace}[gg^\top \text{Hess}\varphi(Y_s^n)] ds. \end{aligned} \quad (3.20)$$

By taking expectation under $\mathbb{P} \otimes \mathbb{P}^m$ we have

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[\varphi(Y_t^n)] + \frac{1}{4} \mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[Z_s^n Z_s^{n\top} \text{Hess}\varphi(Y_s^n)] ds \right] &+ 4n \mathbb{E}\mathbb{E}^m \left[\int_t^T \varphi(Y_s^n) ds \right] \\ &\leq 4 \mathbb{E}\mathbb{E}^m \left[\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^\top f(s, B_s, Y_s^n, Z_s^n) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)] ds \right] \\ &\quad + C \mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[gg^\top \text{Hess}\varphi(Y_s^n)] ds \right]. \end{aligned} \quad (3.21)$$

Taking into account the boundedness of h and $Hess\rho$, we have

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) Hess\varphi(Y_s^n)] ds \right] &\leq 2\mathbb{E}\mathbb{E}^m \left[\int_t^T \langle h(s, B_s, Y_s^n, Z_s^n), \nabla\rho(Y_s^n) \rangle^2 ds \right] \\
&\quad + \mathbb{E}\mathbb{E}^m \left[\int_t^T 2\rho(Y_s^n) \text{trace}[(hh^\top)(s, B_s, Y_s^n, Z_s^n) Hess\rho(Y_s^n)] ds \right] \\
&\leq C\mathbb{E}\mathbb{E}^m \left[\int_t^T |\nabla\rho(Y_s^n)|^2 ds \right] + C\mathbb{E}\mathbb{E}^m \left[\int_t^T 2\rho(Y_s^n) ds \right] \\
&\leq C\mathbb{E}\mathbb{E}^m \left[\int_0^T (d(Y_s^n, D))^2 ds \right].
\end{aligned} \tag{3.22}$$

Applying the same argument to obtain

$$\mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[(gg^\top)(s, B_s, Y_s^n, Z_s^n) Hess\varphi(Y_s^n)] ds \right] \leq C\mathbb{E}\mathbb{E}^m \left[\int_0^T (d(Y_s^n, D))^2 ds \right]. \tag{3.23}$$

Now the inequality $2ab \leq a^2 + b^2$ with $a = (d(Y_s^n, D))^2$ and the boundedness of f yield

$$\begin{aligned}
4(d(Y_s^n, D))^3 |f(s, B_s, Y_s^n, Z_s^n)| &\leq 2(d(Y_s^n, D))^4 + 2(d(Y_s^n, D))^2 |f(s, B_s, Y_s^n, Z_s^n)|^2 \\
&\leq 2\varphi(Y_s^n) + 2C(d(Y_s^n, D))^2.
\end{aligned} \tag{3.24}$$

By plugging the estimate (3.24), (3.23) and (3.22) in (3.21), we obtain thanks to lemma 3.1

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m [\varphi(Y_t^n)] + \frac{1}{4}\mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[Z_s^n Z_s^{n\top} Hess\varphi(Y_s^n)] ds \right] + (4n-2)\mathbb{E}\mathbb{E}^m \left[\int_t^T \varphi(Y_s^n) ds \right] \\
\leq C\mathbb{E}\mathbb{E}^m \left[\int_0^T (d(Y_s^n, D))^2 ds \right] \leq C\left(\frac{1}{n} + \frac{1}{n^2}\right).
\end{aligned} \tag{3.25}$$

Notice also that Hessian of $\varphi(Y_s^n)$ is a positive definite matrix since φ is a convex function, so we get that

$$\sup_{0 \leq t \leq T} \mathbb{E}\mathbb{E}^m [\varphi(Y_t^n)] \leq C\left(\frac{1}{n} + \frac{1}{n^2}\right). \tag{3.26}$$

Moreover, we can deduce from (3.25) that, for every $t \in [0, T]$

$$\mathbb{E}\mathbb{E}^m \left[\int_t^T \text{trace}[Z_s^n Z_s^{n\top} Hess\varphi(Y_s^n)] ds \right] \longrightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.27}$$

On the other hand, taking the supremum over t in the equation (3.20) and by Burkholder-Davis-

Gundy's inequality and the previous calculations it follows that

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m[\sup_{0 \leq t \leq T} \varphi(Y_t^n)] &\leq C\mathbb{E}\mathbb{E}^m[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}\mathbb{E}^m[\int_0^T (d(Y_s^n, D))^2 ds] \\
&\quad + C\mathbb{E}\mathbb{E}^m[\sup_{0 \leq t \leq T} \int_t^T (\rho(Y_s^n) \nabla \rho(Y_s^n))^\top Z_s^n dB_s] \\
&\quad + C\mathbb{E}\mathbb{E}^m[\sup_{0 \leq t \leq T} \int_t^T (\rho(Y_s^n) \nabla \rho(Y_s^n))^\top h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s] \\
&\quad + C\mathbb{E}\mathbb{E}^m[\sup_{0 \leq t \leq T} \int_t^T (\nabla \varphi(Y_s^n))^\top g_s(Y_s^n, Z_s^n) * d\overleftarrow{B}_s] \\
&\leq C\mathbb{E}\mathbb{E}^m[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}[\int_0^T (d(Y_s^n, D))^2 ds] \\
&\quad + C\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds)^{1/2}] \\
&\quad + C\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 \langle \nabla \rho(Y_s^n), h_s(Y_s^n, Z_s^n) \rangle^2 ds)^{1/2}] \\
&\quad + C\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 |\langle \nabla \rho(Y_s^n), g_s(Y_s^n, Z_s^n) \rangle|^2 ds)^{1/2}].
\end{aligned} \tag{3.28}$$

From the boundedness of h and the fact that $\nabla \rho^2(x) = 4\rho(x)$, we have

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 \langle \nabla \rho(Y_s^n), h(s, B_s, Y_s^n, Z_s^n) \rangle^2 ds)^{1/2}] &\leq C\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 \rho(Y_s^n) ds)^{1/2}] \\
&\leq C\mathbb{E}\mathbb{E}^m[\sup_{0 \leq s \leq T} (\varphi(Y_s^n))^{1/2} (\int_0^T \rho(Y_s^n) ds)^{1/2}] \\
&\leq \frac{1}{4}\mathbb{E}\mathbb{E}^m[\sup_{0 \leq s \leq T} \varphi(Y_s^n)] + C^2\mathbb{E}[\int_0^T (d(Y_s^n, D))^2 ds].
\end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 |\langle \nabla \rho(Y_s^n), g(s, B_s, Y_s^n, Z_s^n) \rangle|^2 ds)^{1/2}] \\
&\leq \frac{1}{4}\mathbb{E}\mathbb{E}^m[\sup_{0 \leq s \leq T} \varphi(Y_s^n)] + C^2\mathbb{E}\mathbb{E}^m[\int_0^T (d(Y_s^n, D))^2 ds].
\end{aligned} \tag{3.30}$$

By the Holder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m[(\int_0^T (\rho(Y_s^n))^2 \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds)^{1/2}] &\leq C\mathbb{E}\mathbb{E}^m[\sup_{0 \leq s \leq T} (\varphi(Y_s^n))^{1/2} (\int_0^T \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds)^{1/2}] \\
&\leq \frac{1}{4}\mathbb{E}\mathbb{E}^m[\sup_{0 \leq s \leq T} \varphi(Y_s^n)] + C^2\mathbb{E}\mathbb{E}^m[\int_0^T \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds].
\end{aligned} \tag{3.31}$$

Substituting (3.29), (3.30) and (3.31) in (3.28) leads to

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m[\sup_{0 \leq t \leq T} \varphi(Y_t^n)] &\leq C\mathbb{E}\mathbb{E}^m[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}\mathbb{E}^m[\int_0^T (d(Y_s^n, D))^2 ds] \\
&\quad + C^2\mathbb{E}\mathbb{E}^m[\int_0^T \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds].
\end{aligned} \tag{3.32}$$

Since each term of (3.19) is positive definite and from (3.27), we get

$$\mathbb{E}\mathbb{E}^m \left[\int_0^T \langle \nabla \rho(Y_s^n), Z_s^n \rangle^2 ds \right] \longrightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally, by using (3.26), (3.32) and Lemma 3.1, we get the desired result. \square

Lemma 3.3. *The sequence (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$, i.e.*

$$\mathbb{E}\mathbb{E}^m \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 + \int_0^T \|Z_t^n - Z_t^p\|^2 dt \right] \longrightarrow 0 \quad \text{as } n, p \rightarrow +\infty. \quad (3.33)$$

Proof. For all $n, p \geq 0$, we apply Itô formula to $|Y_t^n - Y_t^p|^2$

$$\begin{aligned} |Y_t^n - Y_t^p|^2 + \int_t^T \|Z_s^n - Z_s^p\|^2 ds &= 2 \int_t^T (Y_s^n - Y_s^p)^\top (f(s, B_s, Y_s^n, Z_s^n) - f(s, B_s, Y_s^p, Z_s^p)) ds \\ &\quad + 2 \int_t^T (Y_s^n - Y_s^p)^\top (h(s, B_s, Y_s^n, Z_s^n) - h(s, B_s, Y_s^p, Z_s^p)) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^n - Y_s^p)^\top (Z_s^n - Z_s^p) dB_s \\ &\quad + \int_t^T (Y_s^n - Y_s^p)^\top (g(s, B_s, Y_s^n, Z_s^n) - g(s, B_s, Y_s^p, Z_s^p)) * dB_s \\ &\quad - 2 \int_t^T \text{trace}[(Z_s^n - Z_s^p)^\top (g(s, B_s, Y_s^n, Z_s^n) - g(s, B_s, Y_s^p, Z_s^p))] ds \\ &\quad + \int_t^T \|h(s, B_s, Y_s^n, Z_s^n) - h(s, B_s, Y_s^p, Z_s^p)\|^2 ds - 2n \int_t^T (Y_s^n - Y_s^p)^\top (Y_s^n - \pi(Y_s^n)) ds \\ &\quad + 2p \int_t^T (Y_s^n - Y_s^p)^\top (Y_s^p - \pi(Y_s^p)) ds. \end{aligned} \quad (3.34)$$

Taking into account the assumptions on g and the inequality $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$, for all $\epsilon > 0$, we have

$$\begin{aligned} -2 \int_t^T \text{trace}[(Z_s^n - Z_s^p)^\top (g(s, B_s, Y_s^n, Z_s^n) - g(s, B_s, Y_s^p, Z_s^p))] ds \\ \leq 2 \int_t^T \|Z_s^n - Z_s^p\| \left(C|Y_s^n - Y_s^p| + \alpha \|Z_s^n - Z_s^p\| \right) ds \\ \leq \epsilon^{-1} \int_t^T |Y_s^n - Y_s^p|^2 ds + (2\alpha + \epsilon) \int_t^T \|Z_s^n - Z_s^p\|^2 ds \end{aligned} \quad (3.35)$$

By the property (2.5), we have

$$\begin{aligned} -2n \int_t^T (Y_s^n - Y_s^p)^\top (Y_s^n - \pi(Y_s^n)) ds + 2p \int_t^T (Y_s^n - Y_s^p)^\top (Y_s^p - \pi(Y_s^p)) ds \\ \leq 2(n+p) \int_t^T (Y_s^n - \pi(Y_s^n))^\top (Y_s^p - \pi(Y_s^p)) ds. \end{aligned} \quad (3.36)$$

Hence, from the Lipschitz continuity on f and h , and taking expectation yields

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[|Y_t^n - Y_t^p|^2] + \mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n - Z_s^p\|^2 ds\right] &\leq 2\mathbb{E}\mathbb{E}^m\left[\int_t^T C(|Y_s^n - Y_s^p|^2 + |Y_s^n - Y_s^p|\|Z_s^n - Z_s^p\|)ds\right] \\ &+ \mathbb{E}\mathbb{E}^m\left[\int_t^T ((C + \epsilon^{-1})|Y_s^n - Y_s^p|^2 + (2\alpha + \epsilon + \beta^2)\|Z_s^n - Z_s^p\|^2)ds\right] \\ &+ 2(n+p)\mathbb{E}\mathbb{E}^m\left[\int_t^T (Y_s^n - \pi(Y_s^n))^\top (Y_s^p - \pi(Y_s^p))ds\right]. \end{aligned} \quad (3.37)$$

For the last term, we need the following lemma whose proof is postponed to the Appendix.

Lemma 3.4. *There exists a constant $C > 0$ such that, for each $n \geq 0$,*

$$\mathbb{E}\mathbb{E}^m\left[\left(n \int_0^T d(Y_s^n, D)ds\right)^2\right] \leq C \quad (3.38)$$

Now we can deduce from the Hölder inequality and Lemma 3.4 that

$$\begin{aligned} n\mathbb{E}\mathbb{E}^m\left[\int_t^T (Y_s^n - \pi(Y_s^n))^\top (Y_s^p - \pi(Y_s^p))ds\right] &\leq n\mathbb{E}\mathbb{E}^m\left[\int_t^T d(Y_s^n, D)d(Y_s^p, D)ds\right] \\ &\leq n\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d(Y_s^p, D) \int_t^T d(Y_s^n, D)ds\right] \\ &\leq \left(\mathbb{E}\mathbb{E}^m\left[\left(n \int_0^T d(Y_s^n, D)ds\right)^2\right]\right)^{1/2} \left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} \\ &\leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2}. \end{aligned} \quad (3.39)$$

Substituting (3.39) in the previous inequality (3.37), we have

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[|Y_t^n - Y_t^p|^2] + (1 - \beta^2 - 2\alpha - C\epsilon)\mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n - Z_s^p\|^2 ds\right] &\leq C(1 + \epsilon^{-1})\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^n - Y_s^p|^2 ds\right] \\ &+ C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2}. \end{aligned}$$

Choosing $1 - \beta^2 - 2\alpha - C\epsilon > 0$, by Gronwall's lemma, we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E}\mathbb{E}^m[|Y_t^n - Y_t^p|^2] \leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2}. \quad (3.40)$$

We deduce similarly

$$\mathbb{E}\mathbb{E}^m\left[\int_0^T \|Z_s^n - Z_s^p\|^2 ds\right] \leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2}. \quad (3.41)$$

Next, by (3.34), the Burkholder-Davis-Gundy inequality and the previous calculations we get

$$\begin{aligned} \mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2\right] &\leq C\mathbb{E}\mathbb{E}^m\left[\int_0^T |Y_s^n - Y_s^p||f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)|ds\right] \\ &+ C\mathbb{E}\mathbb{E}^m\left(\int_0^T |Y_s^n - Y_s^p|^2 \|h(s, B_s, Y_s^n, Z_s^n) - h(s, B_s, Y_s^p, Z_s^p)\|^2 ds\right)^{1/2} + C\mathbb{E}\mathbb{E}^m\left(\int_0^T |Y_s^n - Y_s^p|^2 \|Z_s^n - Z_s^p\|^2 ds\right)^{1/2} \\ &+ C\mathbb{E}\mathbb{E}^m\left(\int_0^T |Y_s^n - Y_s^p|^2 \|g(s, B_s, Y_s^n, Z_s^n) - g(s, B_s, Y_s^p, Z_s^p)\|^2 ds\right)^{1/2} \\ &+ \mathbb{E}\mathbb{E}^m\left[\int_0^T C(|Y_s^n - Y_s^p|^2 + \alpha\|Z_s^n - Z_s^p\|^2)ds\right] + 2(n+p)\mathbb{E}\left[\int_0^T (Y_s^n - \pi(Y_s^n))^\top (Y_s^p - \pi(Y_s^p))ds\right]. \end{aligned}$$

Then, it follows by the Lipschitz Assumption 2.3 on f , g and h and (3.39) that for any $n, p \geq 0$

$$\begin{aligned} \mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2\right] &\leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2} \\ &\quad + C\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \int_0^T \|Z_s^n - Z_s^p\|^2 ds\right)^{1/2} \\ &\leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2} \\ &\quad + C\varepsilon\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2\right) + C\varepsilon^{-1}\mathbb{E}\mathbb{E}^m\left(\int_0^T \|Z_s^n - Z_s^p\|^2 ds\right). \end{aligned}$$

Choosing $1 - C\varepsilon > 0$ and from the inequality (3.41) we conclude that

$$\begin{aligned} \mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2\right] &\leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^p, D)\right]\right)^{1/2} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^2(Y_s^n, D)\right]\right)^{1/2} \\ &\leq C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^4(Y_s^p, D)\right]\right)^{1/4} + C\left(\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq s \leq T} d^4(Y_s^n, D)\right]\right)^{1/4} \rightarrow 0, \end{aligned}$$

as $n, m \rightarrow \infty$, where Lemma 3.2 has been used. \square

Consequently, since for any $n, p \geq 0$ and $0 \leq t \leq T$,

$$\begin{aligned} K_s^n - K_s^p &= Y_0^n - Y_0^p - Y_s^n - Y_s^p - \int_0^s (f(r, Y_r^n, Z_r^n) - f(r, Y_r^p, Z_r^p))dr \\ &\quad - \int_0^s (h(r, Y_r^n, Z_r^n) - h(r, Y_r^p, Z_r^p))d\overleftarrow{W}_r + \int_0^s (Z_r^n - Z_r^p)dB_r. \end{aligned} \quad (3.42)$$

we obtain from (3.33) and Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq s \leq T} |K_s^n - K_s^p|^2\right) \rightarrow 0 \quad \text{as } n, p \rightarrow \infty. \quad (3.43)$$

We have also the following result:

Lemma 3.5. *There exists a \mathcal{F}_s measurable triple processes $(Y_s, Z_s, K_s)_{s \in [0, T]}$ such that*

$$\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2 + \int_0^T |Z_s^n - Z_s|^2 ds + \sup_{0 \leq s \leq T} |K_s^n - K_s|^2\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.44)$$

Moreover, this triple of processes is solution of the following BDSDE in a domain:

$$\left\{ \begin{aligned} (i) \quad &Y_s = \Phi(B_T) + \int_s^T f(r, B_r, Y_r, Z_r)dr + \int_s^T h(r, B_r, Y_r, Z_r)d\overleftarrow{W}_r + K_T - K_s \\ &\quad + \frac{1}{2} \int_t^T g(r, B_r, Y_r, Z_r) * dB_r - \int_s^T Z_r dB_r, \quad \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.}, \quad \forall s \in [t, T] \\ (ii) \quad &Y_s \in \bar{D} \quad \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.} \\ (iii) \quad &\int_0^T (Y_s - v_s(B_s))^\top dK_s \leq 0., \quad \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.}, \\ &\text{for any continuous } \mathcal{F}_s\text{-random function } v : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \bar{D}. \end{aligned} \right. \quad (3.45)$$

Proof. First, we have from (3.33) that (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$ and therefore there exists a unique pair (Y_s, Z_s) of \mathcal{F}_s -measurable processes which valued in $\mathbb{R}^k \times \mathbb{R}^{k \times d}$, satisfying

$$\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2 + \int_0^T |Z_s^n - Z_s|^2 ds\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.46)$$

Similarly, we obtain from (3.44) there exists a \mathcal{F}_s -adapted continuous process $(K_s)_{0 \leq s \leq T}$ (with $K_0 = 0$) such that

$$\mathbb{E}\mathbb{E}^m\left(\sup_{0 \leq s \leq T} |K_s - K_s^n|^2\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore, (A.3) shows that the total variation of K^n is uniformly bounded. Thus, K is also of uniformly bounded variation. Passing to the limit in (3.7), the processes $(Y_s, Z_s, K_s)_{0 \leq s \leq T}$ satisfy

$$\begin{aligned} Y_s &= \Phi(B_T) + \int_s^T f(r, B_r, Y_r, Z_r) dr + \int_s^T h(r, B_r, Y_r, Z_r) d\overleftarrow{W}_r + K_T - K_s \\ &+ \frac{1}{2} \int_t^T g(r, B_r, Y_r, Z_r) * dB_r - \int_s^T Z_r dB_r, \quad \mathbb{P} \otimes \mathbb{P}^m\text{-a.s.}, \quad \forall s \in [t, T] \end{aligned} \quad (3.47)$$

Since we have from Lemma 3.2 that Y_s is in \bar{D} , it remains to check the minimality property for (K_s) , namely i.e., for any continuous \mathcal{F}_s -random function v valued in \bar{D} ,

$$\int_0^T (Y_s - v_s(B_s))^\top dK_s \leq 0.$$

We note that (2.4) gives us

$$\int_0^T (Y_s^n - v_s(B_s))^\top dK_s^n = -n \int_0^T (Y_s^n - v_s(B_s))^\top (Y_s^n - \pi(Y_s^n)) ds \leq 0.$$

Therefore, we will show that we can extract a subsequence such that $\int_0^T (Y_s^n - v_s(B_s))^\top dK_s^n$ converge a.s. to $\int_0^T (Y_s - v_s(B_s))^\top dK_s$. Following the proof of Lemma A.1 in Appendix, we have

$$\begin{aligned} 2\gamma \|K^n\|_{VT} &\leq |\Phi(B_T) - a|^2 + 2 \int_0^T (Y_s^n - a)^\top f(s, B_s, Y_s^n, Z_s^n) ds + 2 \int_0^T (Y_s^n - a)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &- 2 \int_0^T (Y_s^n - a)^\top Z_s^n dB_s + \int_0^T (Y_s^n - a)^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s \\ &- 2 \int_0^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds + \int_0^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds. \end{aligned}$$

Notice that the right hand side tends in probability as n goes to infinity to

$$\begin{aligned} &|\Phi(B_T) - a|^2 + 2 \int_0^T (Y_s - a)^\top f(s, B_s, Y_s, Z_s) ds + 2 \int_0^T (Y_s - a)^\top h(s, B_s, Y_s, Z_s) d\overleftarrow{W}_s - 2 \int_0^T (Y_s - a)^\top Z_s dB_s \\ &+ \int_0^T (Y_s - a)^\top g(s, B_s, Y_s, Z_s) * dB_s - 2 \int_0^T \text{trace}[(Z_s)^\top g(s, B_s, Y_s, Z_s)] ds + \int_0^T \|h(s, B_s, Y_s, Z_s)\|^2 ds. \end{aligned}$$

Thus, there exists a subsequence $(\phi(n))_{n \geq 0}$ such that the convergence is almost surely and $\|K^{\phi(n)}\|_{VT}$ is bounded. Moreover, due to the convergence in \mathbb{L}^2 of $\sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2$ to 0, we can extract a subsequence from $(\phi(n))_{n \geq 0}$ such that $Y^{\phi(\psi(n))}$ converges uniformly to Y . Hence, we apply Lemma

5.8 in [7] and we obtain

$$\int_0^T (Y_s^{\phi(\psi(n))} - v_s(B_s))^{\top} dK_s^{\phi(\psi(n))} \longrightarrow \int_0^T (Y_s - v_s(B_s))^{\top} dK_s \quad a.s. \quad \text{as } n \rightarrow \infty$$

which is the required result. \square

We remind that the purpose of this section is to prove that the penalized solution $(u^n)_n$ is a Cauchy sequence. Thus, by all the calculations done before we obtain:

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_0^T \left(|u^n(s, x) - u^p(s, x)|^2 + |(\nabla u^n)(s, x) - (\nabla u^p)(s, x)|^2 \right) \rho(x) ds dx \\ &= \int_{\mathbb{R}^d} \mathbb{E}^m \mathbb{E} \int_0^T (|Y_s^n - Y_s^p|^2 + \|Z_s^n - Z_s^p\|^2) ds \rho(x) dx \longrightarrow 0. \end{aligned}$$

Therefore $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H}_T , and the limit $u = \lim_{n \rightarrow \infty} u_n$ belongs to \mathcal{H}_T .

Denote $\nu_n(dt, dx) = -n(u_n - \pi(u_n))(t, x) dt dx$ and $\mu_n(dt, dx) = \rho(x) \nu_n(dt, dx)$, then we get

$$\begin{aligned} \mathbb{E} \mathbb{E}^m [|\mu_n|([0, T] \times \mathbb{R}^d)] &= \int_{\mathbb{R}^d} \int_0^T \mathbb{E} \mathbb{E}^m [n|(u_n - \pi(u_n))(s, x)|] ds \rho(x) dx \\ &= \int_{\mathbb{R}^d} \rho(x) \mathbb{E} \mathbb{E}^m \|K^n\|_{V_T} dx \leq C \int_{\mathbb{R}^d} \rho(x) dx < \infty. \end{aligned}$$

It follows that

$$\sup_n |\mu_n|([0, T] \times \mathbb{R}^d) < \infty. \quad (3.48)$$

Moreover by Lemma C.1 (see Appendix C), the sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ is tight. Therefore, there exists a subsequence such that $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to a measure μ . Define $\nu = \rho^{-1} \mu$; μ is a measure such that $\int_0^T \int_{\mathbb{R}^d} \rho(x) |\mu|(dt, dx) < \infty$, and so we have for $\varphi \in \mathcal{D}_T$ with compact support in x ,

$$\int_{\mathbb{R}^d} \int_t^T \varphi d\nu_n = \int_{\mathbb{R}^d} \int_t^T \frac{\varphi}{\rho} d\mu_n \rightarrow \int_{\mathbb{R}^d} \int_t^T \frac{\varphi}{\rho} d\mu = \int_{\mathbb{R}^d} \int_t^T \varphi d\nu.$$

Now passing to the limit in the SPDE (Φ, f_n, g, h) (3.5), we get that that (u, ν) satisfies the reflected SPDE associated to (Φ, f, g, h) , i.e. for every $\varphi \in \mathcal{D}_T$, we have

$$\begin{aligned} & \int_t^T [(u(s, \cdot), \partial_s \varphi(s, \cdot)) + \frac{1}{2} (\nabla u(s, \cdot), \nabla \varphi(s, \cdot))] ds + (u(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) \\ &= \int_t^T [(f(s, \cdot, u^n(s, \cdot), \nabla u(s, \cdot)), \varphi(s, \cdot)) + (g(s, \cdot, u(s, \cdot), \nabla u(s, \cdot)), \varphi(s, \cdot))] ds \\ &+ \int_t^T (h(s, \cdot, u(s, \cdot), \nabla u(s, \cdot)), \varphi(s, \cdot)) d\overleftarrow{W}_s + \int_t^T \int_{\mathbb{R}^d} \varphi(s, x) \nu(ds, dx). \end{aligned} \quad (3.49)$$

We can also deduce the following the probabilistic interpretation (Feymann-Kac's formula) for the measure ν via the nondecreasing process $K^{t,x}$ of the RBDSDE (3.45).

Theorem 3.3. *Let Assumptions 2.2-2.4 hold. Then for every measurable bounded and positive functions φ and ψ ,*

$$\int_{\mathbb{R}^d} \int_t^T \varphi(s, B_s^{-1}) \psi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_s) dK_s^i dx, \quad a.s.. \quad (3.50)$$

Proof.

Since K^n converges to K uniformly in t , the measure dK^n converges to dK weakly in probability. Fix two continuous functions $\varphi, \psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ which have compact support in x and a continuous function with compact support $\theta : \mathbb{R}^d \rightarrow \mathbb{R}^+$, from Bally et al [2] (The proof of Theorem 4), we have (see also Matoussi and Xu [17])

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_t^T \varphi(s, B_{t,s}^{-1}(x)) \psi(s, x) \theta(x) \nu(ds, dx) \\
&= \lim_{n \rightarrow \infty} - \int_{\mathbb{R}^d} \int_t^T \varphi(s, B_{t,s}^{-1}(x)) \psi(s, x) \theta(x) n(u_n - \pi(u_n))(s, x) ds dx \\
&= \lim_{n \rightarrow \infty} - \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_{t,s}(x)) \theta(B_{t,s}(x)) n(u_n - \pi(u_n))(t, B_{t,s}(x)) dt dx \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_{t,s}(x)) \theta(B_{t,s}(x)) dK_s^{n,t,x} dx \\
&= \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_{t,s}(x)) \theta(B_{t,s}(x)) dK_s^{t,x} dx.
\end{aligned}$$

We take $\theta = \theta_R$ to be the regularization of the indicator function of the ball of radius R and pass to the limit with $R \rightarrow \infty$, to get that

$$\int_{\mathbb{R}^d} \int_t^T \varphi(s, B_{t,s}^{-1}(x)) \psi(s, x) \nu(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, B_{t,s}(x)) dK_s^{t,x} dx. \quad (3.51)$$

We know that $dK_s^{t,x} = 1_{\{u \in \partial D\}}(s, B_{t,s}(x)) dK_s^{t,x}$. In (3.51), setting $\psi = 1_{\{u \in \partial D\}}$ yields

$$\int_{\mathbb{R}^d} \int_t^T \varphi(s, B_{t,s}^{-1}(x)) 1_{\{u \in \partial D\}}(s, x) \nu(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, B_{t,s}^{-1}(x)) \nu(ds, dx), \text{ a.s.}$$

Note that the family of functions $A(\omega) = \{(s, x) \rightarrow \phi(s, B_{t,s}^{-1}(x)) : \phi \in C_c^\infty\}$ is an algebra which separates the points (because $x \rightarrow B_{t,s}^{-1}(x)$ is a bijection). Given a compact set G , $A(\omega)$ is dense in $C([0, T] \times G)$. It follows that $J(B_{t,s}^{-1}(x)) 1_{\{u \in \partial D\}}(s, x) \nu(ds, dx) = J(B_{t,s}^{-1}(x)) \nu(ds, dx)$ for almost every ω . While $J(B_{t,s}^{-1}(x)) > 0$ for almost every ω , we get $\nu(ds, dx) = 1_{\{u \in \partial D\}}(s, x) \nu(ds, dx)$, and (3.50) follows.

Then we get easily that $Y_s^{t,x} = u(s, B_{t,s}(x))$ and $Z_s^{t,x} = (\nabla u \sigma)(s, B_{t,s}(x))$, in view of the convergence results for $(Y_s^{n,t,x}, Z_s^{n,t,x})$ and the flow property associated to B . So $u(s, B_{t,s}(x)) = Y_s^{t,x} \in \bar{D}$. Specially for $s = t$, we have $u(t, x) \in \bar{D}$. \square

3.2. Proof of uniqueness for BDSEs

Let (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) be two solutions of the RBDSDE (3.45). Applying the double stochastic Itô's formula extended in Matoussi and Stoica (Corollary 1 and Remark 2 in [16] p.1158)

yields

$$\begin{aligned}
|Y_t^1 - Y_t^2|^2 + \int_t^T \|Z_s^1 - Z_s^2\|^2 ds &= 2 \int_t^T (Y_s^1 - Y_s^2)^\top (f(s, B_s, Y_s^1, Z_s^1) - f(s, B_s, Y_s^2, Z_s^2)) ds \\
&+ 2 \int_t^T (Y_s^1 - Y_s^2)^\top (h(s, B_s, Y_s^1, Z_s^1) - h(s, B_s, Y_s^2, Z_s^2)) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^1 - Y_s^2)^\top (Z_s^1 - Z_s^2) dB_s \\
&+ \int_t^T (Y_s^1 - Y_s^2)^\top (g(s, B_s, Y_s^1, Z_s^1) - g(s, B_s, Y_s^2, Z_s^2)) * dB_s \\
&- 2 \int_t^T \text{trace}[(Z_s^1 - Z_s^2)^\top (g(s, B_s, Y_s^1, Z_s^1) - g(s, B_s, Y_s^2, Z_s^2))] ds \\
&+ \int_t^T \|h(s, B_s, Y_s^1, Z_s^1) - h(s, B_s, Y_s^2, Z_s^2)\|^2 ds + 2 \int_t^T (Y_s^1 - Y_s^2)^\top (dK_s^1 - dK_s^2).
\end{aligned} \tag{3.52}$$

Therefore, under the minimality condition (iv) we have

$$\int_t^T (Y_s^1 - Y_s^2)^\top (dK_s^1 - dK_s^2) \leq 0, \quad \text{for all } t \in [0, T]. \tag{3.53}$$

Then, following the proof of Lemma 3.33 we obtain

$$\mathbb{E}\mathbb{E}^m[|Y_t^1 - Y_t^2|^2] + (1 - \beta^2 - 2\alpha - C\epsilon)\mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \leq C(1 + \epsilon^{-1})\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^1 - Y_s^2|^2 ds\right]$$

Choosing $1 - \beta^2 - 2\alpha - C\epsilon > 0$ and from Gronwall's lemma,

$$\mathbb{E}\mathbb{E}^m[|Y_t^1 - Y_t^2|^2] = 0, \quad \mathbb{E}\mathbb{E}^m\left[\int_0^T \|Z_s^1 - Z_s^2\|^2 ds\right] = 0, \quad 0 \leq t \leq T.$$

Appendix A: A priori estimates

In this section, we provide a priori estimates which are uniform in n on the solutions of (3.7).

Lemma A.1. *There exists a constant $C > 0$, independent of n , such that for all n large enough*

$$\sup_n \mathbb{E}\mathbb{E}^m \left[\sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + \|K^n\|_{VT} \right] \leq C \left[\|\Phi(B_T)\|_2^2 + \int_t^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds \right]. \tag{A.1}$$

Proof. For a given point $a \in D$, that satisfies condition (2.6), we apply generalized Itô's formula to get

$$\begin{aligned}
|Y_t^n - a|^2 + \int_t^T \|Z_s^n\|^2 ds &= |\Phi(B_T) - a|^2 + 2 \int_t^T (Y_s^n - a)^\top f(s, B_s, Y_s^n, Z_s^n) ds \\
&+ 2 \int_t^T (Y_s^n - a)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^n - a)^\top Z_s^n dB_s \\
&+ \int_t^T (Y_s^n - a)^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s - 2 \int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds \\
&+ \int_t^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds - 2n \int_t^T (Y_s^n - a)^\top (Y_s^n - \pi(Y_s^n)) ds.
\end{aligned} \tag{A.2}$$

The stochastic integrals have both zero expectations under $\mathbb{P} \otimes \mathbb{P}^m$ since (Y^n, Z^n) belongs to $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$. We take expectation in (A.2) and we use conditions (2.4) and the Lipschitz Assumption 2.3 in order to obtain

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[|Y_t^n - a|^2] + \mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n\|^2 ds\right] &\leq \mathbb{E}\mathbb{E}^m[|\Phi(B_T) - a|^2] \\ &+ 2C\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^n - a|(|f(s, B_s, a, 0)| + |Y_s^n - a| + \|Z_s^n\|) ds\right] \\ &+ 2\mathbb{E}\mathbb{E}^m\left[\int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds\right] \\ &+ \mathbb{E}\mathbb{E}^m\left[\int_t^T |h(s, B_s, a, 0)|^2 ds\right] + C\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^n - a|^2 ds\right] + \beta^2\mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n\|^2 ds\right]. \end{aligned} \quad (\text{A.3})$$

Taking into account the assumptions on g and the inequality $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$, for all $\epsilon > 0$, we have

$$\begin{aligned} 2\int_t^T \text{trace}[(Z_s^n)^\top (g(s, B_s, Y_s^n, Z_s^n))] ds &\leq 2\int_t^T \|Z_s^n\| \left(|g(s, B_s, a, 0)| + C|Y_s^n - a| + \alpha\|Z_s^n\| \right) ds \\ &\leq \epsilon^{-1} \int_t^T (|g(s, B_s, a, 0)|^2 + |Y_s^n - a|^2) ds + (2\alpha + \epsilon) \int_t^T \|Z_s^n\|^2 ds \end{aligned} \quad (\text{A.4})$$

Plugging estimate (A.4) in (A.3) yields to

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[|Y_t^n - a|^2] + \mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n\|^2 ds\right] &\leq C(\|\Phi(B_T)\|_2^2 + \int_t^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds) \\ &+ C(1 + \epsilon^{-1})\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^n - a|^2 ds\right] + (\beta^2 + 2\alpha + C\epsilon)\mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n\|^2 ds\right]. \end{aligned} \quad (\text{A.5})$$

Thus, if we choose $\epsilon = \frac{1 - \beta^2 - 2\alpha}{2C}$, we have

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[|Y_t^n - a|^2] + \left(\frac{1 - \beta^2 - 2\alpha}{2}\right)\mathbb{E}\mathbb{E}^m\left[\int_t^T \|Z_s^n\|^2 ds\right] &\leq C(\|\Phi(B_T)\|_2^2 + \int_t^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds) \\ &+ C\mathbb{E}\mathbb{E}^m\left[\int_t^T |Y_s^n - a|^2 ds\right]. \end{aligned}$$

Then, it follows from Gronwall's lemma that

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n - a|^2] \leq C(\|\Phi(B_T)\|_2^2 + \int_0^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds).$$

Therefore we can deduce

$$\sup_{0 \leq t \leq T} \mathbb{E}\mathbb{E}^m[|Y_t^n|^2] \leq C(\|\Phi(B_T)\|_2^2 + \int_0^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds),$$

and

$$\mathbb{E}\mathbb{E}^m\left[\int_0^T \|Z_s^n\|^2 ds\right] \leq C(\|\Phi(B_T)\|_2^2 + \int_0^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds).$$

On the other hand, the uniform estimate on Y^n is obtained by taking the supremum over t in the equation (A.2), using the previous calculations and Burkholder-Davis-Gundy inequality. Thus, we get for all $n \geq 0$

$$\mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n - a|^2\right] \leq C \quad \text{and} \quad \mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n|^2\right] \leq C.$$

Finally, the total variation of the process K^n is given by

$$\|K^n\|_{VT} = n \int_0^T |Y_s^n - \pi(Y_s^n)| ds.$$

But from the property (2.6) and the equation (A.2) we have

$$\begin{aligned} 2n \int_t^T \gamma |Y_s^n - \pi(Y_s^n)| ds &\leq 2n \int_t^T |(Y_s^n - a)^\top (Y_s^n - \pi(Y_s^n))| ds \\ &\leq |\Phi(B_T) - a|^2 + 2 \int_t^T (Y_s^n - a)^\top f(s, B_s, Y_s^n, Z_s^n) ds + 2 \int_t^T (Y_s^n - a)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad - 2 \int_t^T (Y_s^n - a)^\top Z_s^n dB_s + \int_t^T (Y_s^n - a)^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s \\ &\quad - 2 \int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds + \int_t^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds \end{aligned}$$

Hence it follows from previous estimates that

$$\mathbb{E}\mathbb{E}^m[\|K^n\|_{VT}] \leq C(\|\Phi(B_T)\|_2^2 + \int_0^T (\|f_s^0\|_{2,2}^2 + \|h_s^0\|_{2,2}^2 + \|g_s^0\|_{2,2}^2) ds),$$

and the proof of Lemma A.1 is complete. \square

Lemma A.2. *There exists a constant $C > 0$, independent of n , such that for all n large enough*

$$\sup_n \mathbb{E}\mathbb{E}^m\left[\sup_{0 \leq t \leq T} |Y_t^n|^4 + \left(\int_0^T \|Z_s^n\|^2 ds\right)^2\right] \leq C \mathbb{E}\mathbb{E}^m\left[|\Phi(B_T)|^4 + \int_0^T (|f_s^0|^4 + |h_s^0|^4 + |g_s^0|^4) ds\right] < \infty. \quad (\text{A.6})$$

(This is still true after slight modification of the proof. Checked!)

Appendix B: Proof of Lemma 3.4.

Let first recall that (Y^n, Z^n) is solution of the BDSDE (3.7) associated to $(\Phi(B_T), f^n, h)$ where $f^n(s, y, z) = f(s, y, z) - n(y - \pi(y))$, for each $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$. Note that, since we have assumed that $0 \in D$, $f_t^n(0, 0) = f_s(0, 0) := f_s^0$. Now, we apply generalized Itô's formula to get

$$\begin{aligned} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + 2n \int_t^T (Y_s^n)^\top (Y_s^n - \pi(Y_s^n)) ds &= |\Phi(B_T)|^2 + 2 \int_t^T (Y_s^n)^\top f(s, B_s, Y_s^n, Z_s^n) ds \\ &\quad + 2 \int_t^T (Y_s^n)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^n)^\top Z_s^n dB_s + \int_t^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds \\ &\quad - \int_t^T (Y_s^n)^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s - 2 \int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds \end{aligned} \quad (\text{B.1})$$

From the property (2.6) and since $0 \in D$ we have

$$\begin{aligned}
2n \int_0^T \gamma |Y_s^n - \pi(Y_s^n)| ds &\leq 2n \int_0^T |(Y_s^n)^\top (Y_s^n - \pi(Y_s^n))| ds \\
&\leq |\Phi(B_T)|^2 + 2 \int_0^T (Y_s^n)^\top f(s, B_s, Y_s^n, Z_s^n) ds + 2 \int_0^T (Y_s^n)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \\
&\quad - 2 \int_0^T (Y_s^n)^\top Z_s^n dB_s + \int_0^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds \\
&\quad + \int_t^T (Y_s^n)^\top g(s, B_s, Y_s^n, Z_s^n) * dB_s - 2 \int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds
\end{aligned}$$

Then, taking the square and the expectation yields

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m \left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds \right)^2 \right] &\leq C\mathbb{E}\mathbb{E}^m [|\Phi(B_T)|^4] + C\mathbb{E}\mathbb{E}^m \left[\left(\int_0^T (Y_s^n)^\top f(s, B_s, Y_s^n, Z_s^n) ds \right)^2 \right] \\
&\quad + C\mathbb{E}\mathbb{E}^m \left[\left(\int_0^T (Y_s^n)^\top h(s, B_s, Y_s^n, Z_s^n) d\overleftarrow{W}_s \right)^2 \right] + C\mathbb{E}\mathbb{E}^m \left[\left(\int_0^T (Y_s^n)^\top Z_s^n dB_s \right)^2 \right] \\
&\quad + C\mathbb{E}\mathbb{E}^m \left[\left(\int_0^T \|h(s, B_s, Y_s^n, Z_s^n)\|^2 ds \right)^2 \right] + C\mathbb{E}\mathbb{E}^m \left[\left(\int_t^T (Y_s^n)^\top g(s, B_s, Y_s^n, Z_s^n) * d\overleftarrow{B}_s \right)^2 \right] \\
&\quad + C\mathbb{E}\mathbb{E}^m \left[\left(\int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds \right)^2 \right].
\end{aligned}$$

By using the isometry property and the boundedness of f , g and h , we obtain

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m \left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds \right)^2 \right] &\leq C\mathbb{E}\mathbb{E}^m [|\Phi(B_T)|^4] + C\mathbb{E}\mathbb{E}^m \left[\left(\int_0^T |Y_s^n|^2 ds \right)^2 \right] \\
&\quad + C\mathbb{E}\mathbb{E}^m \left[\int_0^T |Y_s^n h(s, B_s, Y_s^n, Z_s^n)|^2 ds \right] + C\mathbb{E}\mathbb{E}^m \left[\int_0^T |Y_s^n Z_s^n|^2 ds \right] \\
&\quad + C\mathbb{E}\mathbb{E}^m \left[\int_0^T |Y_s^n g(s, B_s, Y_s^n, Z_s^n)|^2 ds \right] + C\mathbb{E}\mathbb{E}^m \left[\left(\int_t^T \text{trace}[(Z_s^n)^\top g(s, B_s, Y_s^n, Z_s^n)] ds \right)^2 \right] + C.
\end{aligned}$$

Finally, we deduce from Holder inequality and boundedness of h that

$$\mathbb{E}\mathbb{E}^m \left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds \right)^2 \right] \leq C\mathbb{E}\mathbb{E}^m [|\Phi(B_T)|^4] + \int_0^T |Y_s^n|^2 ds + \sup_{0 \leq t \leq T} |Y_t^n|^4 + \left(\int_0^T \|Z_s^n\|^2 ds \right)^2 + C.$$

Thus, from the estimate (A.6) we get the desired result. \square

Appendix C: Proof of the tightness of the sequence $(\mu_n)_{n \in \mathbb{N}}$

Recall first that $\nu_n(dt, dx) = -n(u_n - \pi(u_n))(t, x) dt dx$ and $\mu_n(dt, dx) = \rho(x) \nu_n(dt, dx)$ where u_n is the solution of the SPDEs (3.4).

Lemma C.1. $\mathbb{P} \otimes \mathbb{P}^m$ -a.s. in $\omega \in \Omega$, the sequence of random measure $(\mu_n(\omega, ds, dx))_{n \in \mathbb{N}}$ is tight.

Proof. We shall prove that for every $\epsilon > 0$, there exists some constant K such that

$$\mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} |\mu_n|(ds, dx) \right] \leq \epsilon, \quad \forall n \in \mathbb{N}. \quad (\text{C.1})$$

We first write

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} |\mu_n|(ds, dx) \\
&= \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \left(\mathbf{1}_{\{|B_{0,s}^{-1}(x) - x| \leq K\}} + \mathbf{1}_{\{|B_{0,s}^{-1}(x) - x| \geq K\}} \right) |\mu_n|(ds, dx) \\
&:= I_K^n + L_K^n, \quad \mathbb{P} - a.s.
\end{aligned}$$

Taking expectation yields

$$\mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \mu_n(ds, dx) \right] = \mathbb{E}\mathbb{E}^m [I_K^n] + \mathbb{E}[L_K^n]. \quad (\text{C.2})$$

From Definition 2.2(ii) a), we have $\mu_n(\omega, ds, dx)$ is a $\mathcal{F}_{s,T}^W$ -adapted measure and we know that the inverse of the flow depends only on the noise B . Then, since W and B are mutually independent we get by (3.48) and for $K \geq 2\|b\|_\infty T$, we get

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m [L_K^n] &\leq \mathbb{P} \left(\sup_{0 \leq r \leq T} |B_{0,r}^{-1}(x) - x| \geq K \right) \mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} |\mu_n|(ds, dx) \right] \\
&\leq (C_1 \exp(-C_2 K^2) + C_3 \exp(-C_4 K)) \mathbb{E}[|\mu_n|([0, T] \times \mathbb{R}^d)] \\
&\leq C'_1 \exp(-C_2 K^2) + C'_3 \exp(-C_4 K).
\end{aligned}$$

Finally, for K sufficiently large we obtain

$$\mathbb{E}\mathbb{E}^m [L_K^n] \leq \epsilon.$$

On the other hand, if $|x| \geq 2K$ and $|B_{0,s}^{-1}(x) - x| \leq K$ then $|B_{0,s}^{-1}(x)| \geq K$. Therefore

$$\begin{aligned}
\mathbb{E}\mathbb{E}^m [I_K^n] &\leq \mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|B_{0,s}^{-1}(x)| \geq K\}} \rho(x) |\nu_n|(ds, dx) \right] \\
&= \mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|B_{0,s}^{-1}(x)| \geq K\}} \rho(x) n|u_n - \pi(u_n)|(s, x) ds dx \right]
\end{aligned}$$

which, by the change of variable $y = B_{0,s}^{-1}(x)$, becomes

$$\begin{aligned}
& \mathbb{E}\mathbb{E}^m \left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geq K\}} \rho(B_{0,s}(y)) n|u_n - \pi(u_n)|(s, B_{0,s}(y)) ds dy \right] \\
&\leq \mathbb{E}\mathbb{E}^m \left[\int_{\mathbb{R}^d} \rho(x) \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(B_{0,r}(x)) \right) \|K^n\|_{VT} dx \right] \\
&\leq \left(\mathbb{E}\mathbb{E}^m \left[\int_{\mathbb{R}^d} \|K^n\|_{VT}^2 \rho(x) dx \right] \right)^{1/2} \\
&\quad \left(\mathbb{E}\mathbb{E}^m \left[\int_{\mathbb{R}^d} \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(B_{0,r}(x)) \right)^2 \rho(x) dx \right] \right)^{1/2} \\
&\leq C \left(\mathbb{E}\mathbb{E}^m \left[\int_{\mathbb{R}^d} \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(B_{0,r}(x)) \right)^2 \rho(x) dx \right] \right)^{1/2}.
\end{aligned}$$

where the last inequality is a consequence of (A.3). It is now sufficient to prove that

$$\int_{\mathbb{R}^d} \rho(x)^{-1} \mathbb{E} \mathbb{E}^m \left[\left(\sup_{0 \leq r \leq T} \rho(B_{0,r}(x)) \right)^2 \right] dx < \infty. \quad (\text{C.3})$$

Note that

$$\begin{aligned} \mathbb{E} \mathbb{E}^m \left[\left(\sup_{0 \leq r \leq T} \rho(B_{0,r}(x)) \right)^2 \right] &\leq \left[\mathbb{E} \mathbb{E}^m \left(\sup_{0 \leq r \leq T} |\rho(B_{0,r}(x))| \right)^4 \right]^{1/2} \left[\mathbb{E} \mathbb{E}^m \left(\sup_{0 \leq r \leq T} |J(B_{0,r}(x))| \right)^4 \right]^{1/2} \\ &\leq C \left[\mathbb{E} \mathbb{E}^m \left(\sup_{0 \leq r \leq T} |\rho(B_{0,r}(x))| \right)^4 \right]^{1/2}. \end{aligned}$$

Therefore it is sufficient to prove that:

$$\int_{\mathbb{R}^d} \frac{1}{\rho(x)} \left(\mathbb{E} \mathbb{E}^m \left[\sup_{t \leq r \leq T} |\rho(B_{t,r}(x))|^4 \right] \right)^{1/2} dx < \infty.$$

Since $\rho(x) \leq 1$, we have

$$\begin{aligned} \mathbb{E} \mathbb{E}^m \left[\sup_{t \leq r \leq T} |\rho(B_{t,r}(x))|^4 \right] &\leq \mathbb{E} \mathbb{E}^m \left[\sup_{t \leq r \leq T} |\rho(B_{t,r}(x))|^4 \mathbf{1}_{\left\{ \sup_{t \leq r \leq T} |B_{t,r}(x) - x| \leq \frac{|x|}{2} \right\}} \right] \\ &\quad + \mathbb{P} \left(\sup_{t \leq r \leq T} |B_{t,r}(x) - x| \geq \frac{|x|}{2} \right) \\ &=: A(x) + C(x). \end{aligned}$$

If $\sup_{t \leq r \leq T} |B_{t,r}(x) - x| \leq \frac{|x|}{2}$ then $|B_{t,r}(x)| \geq \frac{|x|}{2}$ and so $|\rho(B_{t,r}(x))| \leq \left(1 + \frac{|x|}{2}\right)^{-p}$. Thus we have that $A(x) \leq \left(1 + \frac{|x|}{2}\right)^{-4p}$ and so $\int_{\mathbb{R}^d} (1 + |x|)^p A(x)^{1/2} dx < \infty$. On the other hand, if $|x| \geq 4\|b\|_{\infty}T$, then (the same argument as in the existence proof step 2 of Theorem 4 in [2] for the Itô integral with respect to the Brownian motion)

$$\begin{aligned} C(x) &\leq \mathbb{P} \left(\sup_{t \leq s \leq T} \left| \int_0^s dW_r \right| \geq \frac{|x|}{8} \right) \\ &\leq C_1 \exp(-C_2 |x|^2) \end{aligned}$$

and so $\int_{\mathbb{R}^d} (1 + |x|)^p C(x)^{1/2} dx < \infty$. □

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